MATH 121A Prep: Subspaces

Facts to Know:

Definition: Let V be a *non-empty* subset of $\mathbb{R}^n$. V is called a Subspace of $\mathbb{R}^n$ if:

(1) For every $\vec{v_1}, \vec{v_2} \in V$ we have $\vec{v_1} + \vec{v_2} \in V$ closed under addition

(2) For every $\vec{v} \in V$ and $c \in \mathbb{R}$ we have $c\vec{v} \in V$ closed under scalar multiplication

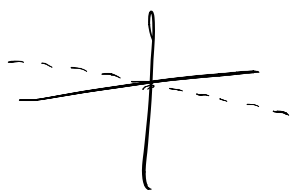
$\vec{v} \in V$, $0 \in \mathbb{R}$. Then $0\vec{v} = \vec{0} \in V$

Examples:

1. Show that the line $V = \{(x, y) \in \mathbb{R}^2 : x + 2y = 0\}$ is a subspace of $\mathbb{R}^2$.

line $y = -\frac{1}{2}x$

Non-empty: $\vec{0} \in V$ $\quad 0 + 2(0) = 0$ ✓



(1) $\vec{v}_1 = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ $\vec{v}_2 = \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \in V$ so $x_1 + 2y_1 = 0$, $x_2 + 2y_2 = 0$

$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} x_1 + x_2 \\ y_1 + y_2 \end{bmatrix}$ WTS $\vec{v}_1 + \vec{v}_2 \in V$, so

$(x_1 + x_2) + 2(y_1 + y_2) = \underbrace{(x_1 + 2y_1)}_{=0} + \underbrace{(x_2 + 2y_2)}_{=0} = 0$

$\vec{v}_1 + \vec{v}_2 \in V$

(2) $\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}$, $c \in \mathbb{R}$ $\quad \vec{v} \in V$ so $x + 2y = 0$

$c\vec{v} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ $\quad cx + 2(cy) = cx + 2cy = c(x + 2y) = c \cdot 0 = 0$ ✓

2. Show that the plane $V = \{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1\}$ is NOT a subspace of $\mathbb{R}^3$.

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \in V$

$\vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in V$

$\vec{v}_1 + \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \notin V$ since

$1 + 1 + 0 = 2 \neq 1$

NOT a subspace

$\vec{0} \notin V$ since $0 + 0 + 0 \neq 1$

$\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

3. Let $A = \begin{bmatrix} 1 & 1 & -2 \\ 2 & 3 & 0 \end{bmatrix}$ and let $V = \{\vec{v} \in \mathbb{R}^3 : A\vec{v} = \vec{0}\}$. Show that V is a subspace of $\mathbb{R}^3$ and find a vector that is in V and a vector that is not in V.

Non-empty: $A\vec{0} = \vec{0}$ so $\vec{0} \in V$ ✓

(1) $\vec{v}_1, \vec{v}_2 \in V$ then $A\vec{v}_1 = A\vec{v}_2 = 0$

$$A(\underbrace{\vec{v}_1 + \vec{v}_2}_{\in V}) = A\vec{v}_1 + A\vec{v}_2 = \vec{0} + \vec{0} = \vec{0} \checkmark$$

(2) $c \in \mathbb{R}$, $\vec{v} \in V$ then $A\vec{v} = \vec{0}$

$$A(\underbrace{c\vec{v}}_{\in V}) = cA\vec{v} = c\vec{0} = \vec{0} \checkmark$$

So subspace of $\mathbb{R}^3$!

$$A\vec{v} = \vec{0} \quad \left[\begin{array}{ccc|c} 1 & 1 & -2 & 0 \\ 2 & 3 & 0 & 0 \end{array}\right] \xrightarrow{R2 = R2 - 2R1} \left[\begin{array}{ccc|c} 1 & \boxed{1} & -2 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right]$$

$$\xrightarrow{R1 = R1 - R2} \left[\begin{array}{ccc|c} 1 & 0 & -6 & 0 \\ 0 & 1 & 4 & 0 \end{array}\right] \quad \begin{array}{l} x_1 - 6x_3 = 0 \to x_1 = 6x_3 \\ x_2 + 4x_3 = 0 \to x_2 = -4x_3 \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_3 \\ -4x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 6 \\ -4 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} 6 \\ -4 \\ 1 \end{bmatrix} \in V$$

$$\begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix} \notin V$$